Semiclassical Nonlinear Approach for Mesoscopic Circuit

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Abstract— *Based on energy conservation considerations we study the nonlinear dynamic behavior of a quantum mesoscopic circuit, which is characterized through parameters of inductance and capacitance. Nonlinearity is given by the initial conditions of magnetic flux and discreteness charge which oscillate in the interval [-* α_0 *, +* α_0 *], being* α_0 *the magnetic flux normalized by* $\hbar / q_e = \phi_0$ *. This LC circuit with quantized electric charge is excited by energy battery that can produce an electrical discreteness charge on the capacitor. The dynamics of the mesoscopic circuit is highly nonlinear. Our results show for the magnetic flux a nearly square wave with an elongated period when compared with the linear case and a train of narrow pulses for the discrete charge.*

Keywords— LC circuit, mesoscopic, discreteness charge.

I. INTRODUCTION

In a series of articles Li and Chen [1, 2] and Flores et al [3, 4, 5, 6, 7], have developed a theory of quantum electrical systems, based on a treating such systems as quantum LC circuits; that is, electrical systems described by two parameters: an inductance L, and a capacitance C. Such quantum theory of circuits is expected to apply when the transport dimension becomes comparable with the charge carrier coherence length, taking into account both the quantum mechanical properties of the electron system, and also the discrete nature of electric charge. Now, we propose a semiclassical theory of quantum electrical circuits, to obtain useful predictions of the theory from very simple calculations of energy consideration , and to push the circuit analogy one step further, generalizing the Heisenberg equations of motion .

The semiclassical theory of quantum LC circuits [1] starts from the quantum Hamiltonian of the LC circuit [1-7, 8, 9]. The resulting equations become

$$H = \frac{2\hbar^2}{q_e^2 L}\sin\left(\frac{q_e\phi}{2\hbar}\right) + \frac{Q^2}{2C} \qquad (1)$$

$$\frac{\partial H}{\partial Q} = \frac{Q}{C} = -\dot{\phi}, \quad \frac{\partial H}{\partial \phi} = \dot{Q} = \frac{\hbar}{q_e L}\sin(\phi/\phi_0) \qquad (2)$$

$$\ddot{\phi} + \frac{\phi_0}{LC}\sin(\phi/\phi_0) \qquad (3)$$

which we can put as

$$\ddot{\alpha} + \omega_0^2 \sin(\alpha) = 0 \qquad (3')$$

where $\omega_0^2 = 1/LC$, $\alpha = \phi/\phi_0$. The equations above are considered, mathematically, as classical equations, but they include quantum effects, the quantized nature of electric charge through the parameter $\hbar / q_e = \phi_0$. These equations are highly nonlinear, but they reduce to the usual equations of the LC circuit, in the discrete charge limit, $q_e \to 0$. in this paper we are interested in the highly nonlinear behavior of the LC circuit excited by a battery source with initial conditions $\phi_0 \neq 0$ and $d(\phi/\phi_0)/dt \neq 0$, and $\phi/\phi_0 \approx \pi$.

We will resolve the nonlinear equation through energy considerations because our starting point is the Hamiltonian equation (1)

II. DIFFERENTIAL EQUATION FROM ENERGY CONSIDERATIONS

At the initial instance, the energy of the mesoscopic circuit is the sum of the kinetic and potential energy, from (4) we obtain :

$$E_0 = \dot{\alpha}_0^2 + E_p \sin^2(\alpha_0/2) \qquad (4)$$

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with $E_0 = E_{B=0} + E_B$, $E_p = 4\omega_0^2 = 4/LC$ which has been normalized and $\alpha = \phi/\phi_0$ $\dot{\alpha} = d(\phi/\phi_0)/dt$, $\hbar / q_e = \phi_0$. Because of the assumed lossless system, the initial energy is preserved for all instances:

$$E_0 = \dot{\alpha}^2(t) + E_p \sin^2(\alpha(t)/2) \quad (5)$$

Here, $E_p/2 = 2\omega_0^2$ is the maximum possible (normalized) potential energy of the quantum circuit, being attained when $\dot{\alpha} = 0$.

Equation (5) refers to the conservation of the total mechanical energy, to integrate and apply the initial conditions given α_0, $\dot{\alpha}_0$, we obtain

$$(\frac{d\alpha}{dt})^2 = 2\omega_0^2\left[\cos(\alpha(t)) - \cos(\alpha_0(t))\right] + \dot{\alpha}_0 \quad (6)$$

The analogy between a quantum LC circuit with discrete load and nonlinear pendulum is straightforward because both systems are modeled by the same equations (3'), (6).

These equations are similar to that obtained for nonlinear pendulum, The simple gravity pendulum is a famous case study in classical mechanics that leads to a nonlinear differential equation of second order like, equation (3'). A solution of the differential equation (6) is based on Jacobi elliptic integrals has been well known. There exists a great number of papers and textbooks dealing with pendulums, the reader is referred to [10],

So following [10-13] and after several steps algebraic we can obtain the normalized magnetic flux, in equation (6) (see also appendix A of [14]), where we have incorporated the initial condition of $d\alpha/dt|_{t=0}$.

$$\alpha(t) = \frac{\phi}{\phi_0} = 2\sin^{-1}\left[u_o\sqrt{k}\,sn(F(\sin^{-1}(\frac{1}{u_0}), u_0^2k) \pm \omega_0 t, u_0^2 k\right] \quad (7)$$

where

$$u_0^2 = 1 + \frac{\dot{\alpha}_0^2}{4k\omega_0^2}, \quad k = \sin^2(\frac{\alpha_0}{2}), \quad \alpha_0 = (\phi/\phi_0)|_{t=0} \quad (8)$$

In particular, if the angular velocity is zero, $\dot{\alpha}_0 = 0$, then $u_0^2 = 1$ therefore equation (22) reproduces the known formula for the nonlinear pendulum, [12, 13].

$$\alpha(t) = 2\sin^{-1}\left[\sqrt{k}\,sn(F(\sin^{-1}(1), k) \pm \omega_0 t, k\right] \quad (9)$$

However $F(\frac{\pi}{2}, k) = K(k)$ is the first class elliptical integral so

$$\alpha(t) = 2\sin^{-1}\left[\sqrt{k}\,sn(K(k) \pm \omega_0 t, k)\right] \quad (10)$$

Generally the first equation of (8) cannot have any value, it must be limited because for a LC circuit the amplitude $\alpha \in [-\alpha_0, \alpha_0]$ so $0 < k < 1$, and

$$u_0^2 k = k + \frac{\dot{\alpha}_0^2}{4\omega_0^2} = k_{eff} \quad (11)$$

which also k_{eff} must be $k_{eff} \le 1$. If we put $k_{eff} \approx 1$, then

$$\frac{\dot{\alpha}_0^2}{4\omega_0^2} < 1-k \quad \text{or} \quad \dot{\alpha}_0 < 2\omega_0\sqrt{1-k} \rightarrow \dot{\alpha}_0 \ll 2\omega_0. \quad (12)$$

The period T is the time required to complete a cycle, in this case, the oscillation period T is four times the time taken from $\alpha = 0, (v = 0)$, $\alpha = \alpha_0, (v = 1)$ so

$$\omega_0/\omega = 4K(k) \quad (13)$$

the parameter k is related to $\dot{\alpha}_0$ and ω_0 as

$$k = \frac{1}{2\omega_0}\dot{\alpha}_0 \quad (14)$$

III. NUMERICAL RESULTS

Here we show the result of calculation of ω as a function of $\dot{\alpha}_0$. For simplicity, we assume that $\omega_0 = 1$. For small values of $\alpha = \phi/\phi_0$, the nonlinear equation (10) becomes equal to a resonant circuit where the waveform is completely sinusoidal.

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In figure 1, we show $\alpha = \phi / \phi_0$ versus t/T for $\alpha_0 = \pi/2$. having as parameter $\dot{\alpha}_0 = (\dot{\phi}) / \phi_0 (t=0) = -0.5$, dashed point-line, 0, solid line and 0,5 dashed line respectively. The magnetic flux is normalized by $\phi_0 = \hbar / q_e$, that is $\alpha = \phi / \phi_0$.

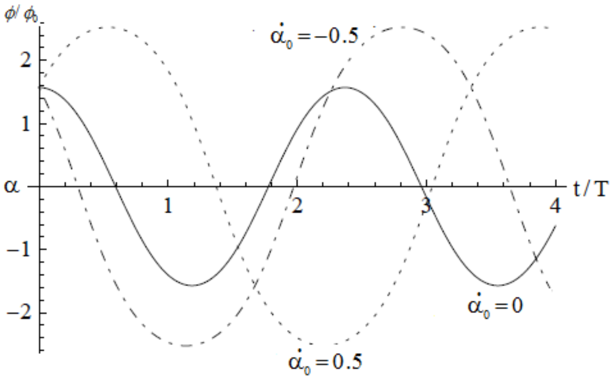

Fig.1: Plot of $\alpha = \phi / \phi_0$ *versus* t/T *for* $\alpha_0 = \pi/2$. *having as parameter* $\dot{\alpha}_0 = (\dot{\phi}) / \phi_0 (t=0) = -0.5$, *dashed point-line, 0, solid line and 0,5 dashed line respectively.*

At small oscillations, the energy E_0, equation (4) of the circuit is small compared to the maximum possible potential energy $E_p = 4\omega_0^2$. This leads to a modulus close to zero, for which the Jacobi elliptic functions can be replaced by trigonometric functions

The period of oscillation of the mesoscopic circuit is constant and independent of the initial angular displacement for values of $\alpha_0 = \pi/2$, and $(d\alpha/dt)_0 = 0$ as shown in Figure 1, solid line. As the initial magnetic flux increases (and its initial time derivative), the oscillation period increases considerably when $\alpha_0 \approx \pi$. That is, the oscillation becomes disharmonious. For this reason, once the movement becomes dissonant or anharmonic significantly, the period (frequency) no longer remains constant but lengthens increasingly (shortens) with increasing amplitude of $\dot{\alpha}_0$ as is shown in figures 2 and 3. Figure 3 is an enlarged version of Figure 2 near the value $\dot{\alpha}_0 = 2$ corresponding to equation (23) with $k = 1$.

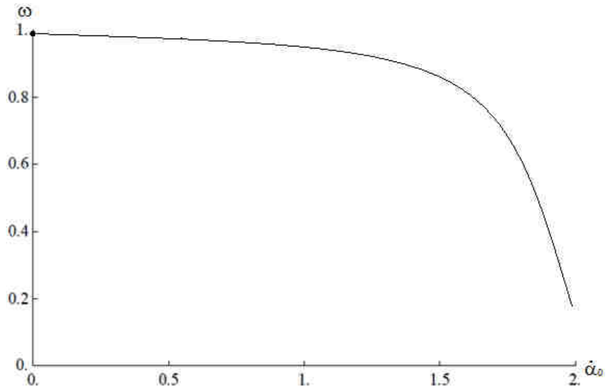

Fig.2: Plot of ω *vs* $\dot{\alpha}_0$ *for* $\omega_0 = 1$. *The angular frequency reduces to zero at* $\dot{\alpha}_0 = 2$.

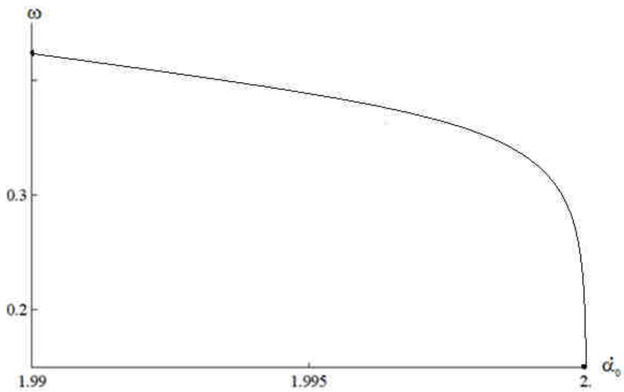

Fig.3: Plot of ω *vs* $\dot{\alpha}_0$ *for* $\omega_0 = 1$, $1.99 \le \dot{\alpha}_0 \le 2$. *The angular frequency reduces to zero at* $\dot{\alpha}_0 = 2$.

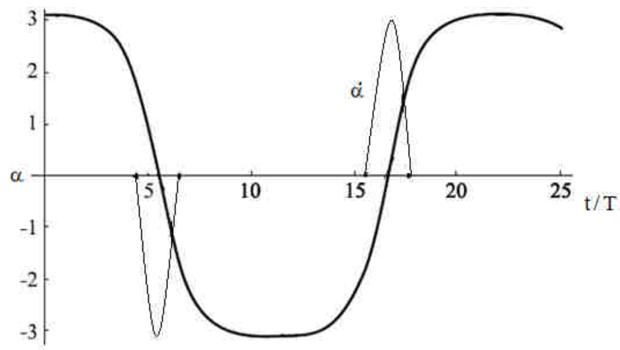

Fig.4: The amplitude α *and the angular velocity* $\dot{\alpha}$ *are also periodic functions for* $\alpha_0 = 0.995\pi$, *:* $E_B = 0.5$. *At microwave frequency* $t/T = \omega_0 t \times 2\pi \times 10^{-8}$, $\alpha(t) = \phi / \phi_0$ *is nearly a square waveform.*

Figure 4, shows the amplitude α and the angular velocity $\dot{\alpha}$. They are also periodic functions for $\alpha_0 = 0.995\pi$, with $E_B = 0.5$, but α is almost a square wave and $\dot{\alpha}$ is a train of narrow pulses. Here, the dynamics of the mesoscopic circuit is highly nonlinear. Here, $\dot{\alpha}$ is proportional to the discreteness charge Q of the quantum circuit

$$\frac{Q}{C} = -\dot{\phi} \rightarrow \frac{Q}{C\phi_0} = -\dot{\alpha} \tag{24}$$

Waveforms of $d\alpha / dt$ are appropriate for design of bits sequence for digital systems at high frequency.

IV. CONCLUSIONS

Based on energy conservation considerations we study the nonlinear dynamic behavior of a quantum mesoscopic circuit, which is characterized through parameters of inductance and capacitance. Nonlinearity is given by the initial conditions of magnetic flux and discreteness charge which oscillate in the interval [$-\alpha_0$, $+\alpha_0$], being α_0 the magnetic flux normalized by $\hbar / q_e = \phi_0$ and it is close to $\pm\pi$. This LC circuit with quantized electric charge is excited by energy battery that can produce an electrical discreteness charge in LC in the form of narrow pulses. The results presented also intend to bring the student of physics and engineering, the introduction of elliptic integrals and motivate the search for new alternatives in the design of electronic circuits besides to solve physical and applied mathematical problems.

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